

Effects of Vacuum Fluctuation Suppression on Atomic Decay Rates

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Abstract

The use of atomic decay rates as a probe of sub-vacuum phenomena will be studied. Because electromagnetic vacuum fluctuations are essential for radiative decay of excited atomic states, decay rates can serve as a measure of the suppression of vacuum fluctuation in non-classical states, such as squeezed vacuum states. In such states the renormalized expectation value of the square of the electric field or the energy density can be periodically negative, representing suppression of vacuum fluctuations. We explore the extent to which atomic decays can be used to measure the mean squared electric field or energy density. We consider a scheme in which atoms in an excited state transit a closed cavity whose lowest mode contains photons in a non-classical state. The change in the decay probability of the atom in the cavity due to the non-classical state can, under certain circumstances, serve as a measure of the mean squared electric field or energy density in the cavity. We derive a quantum inequality bound on the decrease in this probability. We also show that the decrease in decay rate can sometimes be a measure of negative energy density or negative squared electric field. We make some estimates of the magnitude of this effect, which indicate that an experimental test might be possible.

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I. INTRODUCTION

It has been known for some time that negative energy densities and fluxes are a generic prediction of quantum field theory [1]. States which involve negative energy, such as the Casimir vacuum state and squeezed states of light, have even been produced in the laboratory [2, 3]. Negative energy density may be viewed as an example of a sub-vacuum phenomenon, whereby the vacuum fluctuations are suppressed below their level in the Minkowski vacuum state. The experiments which have been done measure an indirect effect, such as a Casimir force or a change in photon counting statistics, not the energy density itself. This raises the question of whether more direct detection of negative energy density or related effects is possible.

The gravitational effects are far too small to be feasible in a laboratory experiment. Furthermore, the magnitude and duration of negative energy densities and fluxes are constrained by quantum inequalities [4, 5, 6, 7, 8]. These are constraints derivable directly from quantum field theory, which yield an inverse relation between the magnitude of the negative energy density and its duration. Marecki [9] proved quantum inequality-type bounds on the magnitude and duration of squeezing in quantum optics experiments involving squeezed states of light. These constraints make the search for local measurements of sub-vacuum effects more challenging.

Nonetheless, there seems to be no barrier in principle to constructing a negative energy detector which relies solely upon electromagnetic interactions. One model for such a detector was proposed by Ford, Grove, and Ottewill [10], who analyzed a system of atomic spins, placed in an external magnetic field and coupled to a quantized electromagnetic field. When the system is coupled to a non-classical state of photons, such as a squeezed vacuum state, there can be a transient increase in the average magnetic moment, as compared to when the quantized field is in the vacuum state. This can be viewed as a suppression of a depolarizing effect of vacuum fluctuations on the spins, resulting in momentary “re-polarization”. In general, this change is not directly correlated with energy density. However, for a certain ranges of parameters of the system, the change in magnetic moment is in phase with the periods of negative energy density. Under these circumstances, this spin system represents a non-gravitational negative energy detector. Unfortunately, the fractional change in the mean magnetic moment is quite small unless the photon energies approach the γ -ray range. Consequently, it is questionable that one could find a way to detect the extremely rapid changes in the mean magnetic moment.

Other models have been treated by Davies and Ottewill [11], who studied the detection of negative energy fluxes using various types of switched monopole particle detectors, building on earlier work by Grove [12]. Marecki and Szpak [13] modeled spontaneous light emission from two-level atoms coupled to a quantized electromagnetic field. They derived a Volterra-type equation which controls the time evolution of the amplitude of the excited state.

In this paper, we follow the spirit of these earlier works and look for an indirect “tracker” which might lead to the detection of negative energy or vacuum fluctuation suppression-type effects. It is well-known that electromagnetic vacuum fluctuations are essential for the spontaneous decay of excited states of atoms. Without the coupling to the quantized radiation field, all energy levels of an atom would be eigenstates of the Hamiltonian, and hence stable. This suggests that a suppression of the usual vacuum fluctuations could increase the lifetime of an excited state in a way which might be observable.

We consider a model of a two-level excited atom interacting with a quantized electro-

magnetic field, using first-order perturbation theory. The field is confined to a closed cavity with two of its dimensions much larger than the other one. This is so that we can arrange the transit time of the atom through the cavity to be small compared to the light travel time across the larger two dimensions of the cavity. Our idea is to prepare the cavity field in a non-classical state, such as a squeezed vacuum, and fire the excited atom through the cavity along its shortest dimension, before the state of the field has time to change very much. We want the atom to interact with the field during the period when the renormalized expectation value of the square of the electric field, $\langle E^2 \rangle$, is negative. The purpose is to see whether this will suppress the de-excitation probability of the atom, compared to its vacuum value, in a way that will be correlated with the periods of $\langle E^2 \rangle < 0$. If so, our system would function as a negative E^2 detector. We work primarily with cavity modes, for which $\langle E^2 \rangle \neq \langle B^2 \rangle$, so periods of $\langle E^2 \rangle < 0$ do not generally correspond to periods of negative energy density. However, under certain conditions, our model will also serve to measure the energy density.

For most treatments of this type in quantum optics, the Jaynes-Cummings model for the interaction between the atom and the field is used. This model makes use of the rotating wave approximation, which effectively ignores terms with rapidly oscillating exponentials [14]. We explicitly do not make this approximation, because the effects we are interested in are highly transient ones, and depend on these terms.

It also should be mentioned that any scheme which is designed so that a particle or observer interacts only with negative energy in flat spacetime can be ruled out by the quantum inequalities and the averaged weak energy condition, which is known to hold in Minkowski spacetime. In a realistic version of our atom-cavity system, the atom would have to pass through holes cut in the cavity walls. Even in the event that the atom passed through net negative energy while passing through the cavity, edge effects from the holes will contribute enough positive energy to satisfy the averaged weak energy condition. This effect has been discussed recently by Graham and Olum [15], and by Fewster, Olum, and Pfenning [16].

In our case, we are comparing the de-excitation probability of the atom in an excited state of the cavity to when there is only vacuum in the cavity. In particular, we are interested in situations where this probability is suppressed relative to its vacuum value, and where the periods of suppression are in phase with periods of $\langle E^2 \rangle < 0$. We will be concerned with changes in $\langle E^2 \rangle$ or the energy density due to changes in the quantum state in the cavity. In this case, edge effects due to the holes in the cavity will cancel out. It is possible for the difference in the net energy seen by an observer to be negative, as was shown by Borde, Ford, and Roman [17].

Throughout this paper, we will regard $\langle E^2 \rangle$ and the energy density as being set to zero in the vacuum state of the cavity. Thus we are concerned only with changes due to changing the quantum state of the cavity, and not with Casimir-type effects due to the geometry of the cavity. It is well known that the presence of boundaries can make the energy density or $\langle E^2 \rangle$ smaller than in empty space vacuum state. However, this effect on decay rates of atoms is difficult to distinguish from the effects of the changes in mode structure, such as the change from a continuous to a discrete spectrum. The effects of cavity geometry on atomic decays has been extensively studied in recent decades, with an early treatment given by Babiker and Barton [18]. For a recent review, see for example Ref. [19].

The outline of this paper is as follows: In Sect. II, we develop some formalism for describing the interaction of cavity modes with an atom. This is further developed in Sect. III,

where we obtain detailed expressions for the de-excitation probability for an atom traversing a cavity with one mode excited. In Sect. IV, we make some numerical estimates of the size of the effect and discuss the feasibility of observing it. Our results are summarized in Sect. V. In the Appendix, we derive a quantum inequality-type bound on the de-excitation probability. Unless stated otherwise, we work in Lorentz-Heaviside units where $\hbar = c = 1$.

II. ATOM-CAVITY INTERACTION

Consider a two-level atom with states $|\psi_1\rangle$ and $|\psi_2\rangle$. If the atom passes through a cavity containing a quantized radiation field, then the final state of the system will in general be an entangled one. First, consider the case where the initial photon state in the cavity is a single-mode number eigenstate, $|\gamma_i\rangle = |n\rangle$, and the atom is in the state $|\psi_1\rangle$. Then the initial state of the system is

$$|\Psi_i\rangle = |n\rangle |\psi_1\rangle. \quad (1)$$

If we consider the case where the interaction changes the photon number by at most one, then the most general final state is of the entangled form:

$$|\Psi_f\rangle = B_1 |n\rangle |\psi_1\rangle + B_2 |n+1\rangle |\psi_2\rangle + B_3 |n-1\rangle |\psi_1\rangle + B_4 |n+1\rangle |\psi_1\rangle + B_5 |n\rangle |\psi_2\rangle + B_6 |n-1\rangle |\psi_2\rangle. \quad (2)$$

Suppose we want the probability of finding the atom in state $|\psi_2\rangle$, irrespective of the photon state. Then we should project $|\Psi_f\rangle$ onto the subspace where $|\psi_{atom}\rangle = |\psi_2\rangle$:

$$\langle\psi_2|\Psi_f\rangle = B_2 |n+1\rangle + B_5 |n\rangle + B_6 |n-1\rangle. \quad (3)$$

However, $B_5 = 0$, because the interaction term only connects photon number states which differ by ± 1 .

The interaction Hamiltonian for the atom and the field in the cavity, in the dipole approximation, is given by:

$$H' = -\mathbf{d} \cdot \mathbf{E}_S \quad (4)$$

where \mathbf{d} is the dipole moment of the atom and \mathbf{E}_S is the Schrödinger picture electric field operator for the quantized cavity field, evaluated at the atom's position. The dipole approximation assumes that we can ignore the variation of the field across the size of the atom. The Schrödinger picture electric field is related to the Heisenberg field $\mathbf{E}(\mathbf{x}, t)$ by $\mathbf{E}_S(\mathbf{x}) = \mathbf{E}(\mathbf{x}, 0)$. The Heisenberg field has the mode expansion

$$\mathbf{E}(\mathbf{x}, t) = \sum_{k\lambda} [a_{k\lambda} \hat{\mathbf{e}}_{k\lambda} f_{k\lambda}(\mathbf{x}) e^{-i\omega t} + a_{k\lambda}^\dagger \hat{\mathbf{e}}_{k\lambda} f_{k\lambda}(\mathbf{x}) e^{i\omega t}]. \quad (5)$$

Here $f_{k\lambda}(\mathbf{x})$ is the spatial part of the mode function, which we take to be real, and $\hat{\mathbf{e}}_{k\lambda}$ is a linear polarization vector.

First order perturbation theory yields

$$\begin{aligned} B_2 &= -i \int_{t_0}^{t_1} dt' \langle n+1, \psi_2 | H' | n, \psi_1 \rangle e^{i(\omega - \Delta\varepsilon)t'} \\ &= i\sqrt{n+1} \int_{t_0}^{t_1} dt' \hat{\mathbf{e}}_{k\lambda} \cdot \langle \psi_2 | \mathbf{d} | \psi_1 \rangle f_{k\lambda}(\mathbf{x}(t')) e^{i(\omega - \Delta\varepsilon)t'} \end{aligned} \quad (6)$$

$$\begin{aligned} B_6 &= -i \int_{t_0}^{t_1} dt' \langle n-1, \psi_2 | H' | n, \psi_1 \rangle e^{-i(\omega + \Delta\varepsilon)t'} \\ &= i\sqrt{n} \int_{t_0}^{t_1} dt' \hat{\mathbf{e}}_{k\lambda} \cdot \langle \psi_2 | \mathbf{d} | \psi_1 \rangle f_{k\lambda}(\mathbf{x}(t')) e^{-i(\omega + \Delta\varepsilon)t'}. \end{aligned} \quad (7)$$

We assume that $|\psi_1\rangle$ is the higher of the two energy states, and let $\Delta\varepsilon > 0$ be the energy difference between the two atomic states. The probability of finding the atom in $|\psi_2\rangle$ is

$$P_2 = |\langle\psi_2|\Psi_f\rangle|^2 = |B_2|^2 + |B_6|^2. \quad (8)$$

Note that the contribution of B_6 would be ignored in the rotating wave approximation.

Let us now consider the case of a general one-mode initial photon state

$$|\gamma_i\rangle = \sum_{n=0}^{\infty} c_n |n\rangle. \quad (9)$$

The most general final state of the atom-cavity system is

$$|\Psi_f\rangle = \sum_{n=0}^{\infty} (A_{1n}|n\rangle|\psi_1\rangle + A_{2n}|n\rangle|\psi_2\rangle). \quad (10)$$

Now the projection of $|\Psi_f\rangle$ onto $|\psi_2\rangle$ is the vector in the photon state space given by

$$\langle\psi_2|\Psi_f\rangle = \sum_{n=0}^{\infty} A_{2n}|n\rangle, \quad (11)$$

and the probability of finding $|\psi_2\rangle$ is

$$P_2 = |\langle\psi_2|\Psi_f\rangle|^2 = \sum_{n=0}^{\infty} |A_{2n}|^2. \quad (12)$$

Let T be the transition matrix, so that $T_{fi} = \langle f|T|i\rangle$ is the amplitude to make a transition from state $|i\rangle$ to state $|f\rangle$. Then we have that

$$\begin{aligned} A_{2m} &= \langle m\psi_2|T|\gamma_i\psi_1\rangle \\ &= \sum_{n=0}^{\infty} c_n \langle m\psi_2|T|n\psi_1\rangle. \end{aligned} \quad (13)$$

This latter matrix element between energy eigenstates is then given in first-order perturbation theory by an integral of a matrix element of H' .

III. DE-EXCITATION PROBABILITY FOR AN ATOM IN A CAVITY

Consider an atom passing through a rectangular cavity with dimensions a , b , and d aligned along the x , y , z axes respectively, as shown in Fig. 1. We will assume that $b < a < d$, with $b \ll d$, and that the velocity of the atom is parallel to the b -dimension of the cavity, $\mathbf{v} = v\hat{y}$. We will also assume that $v \ll 1$, so that we may ignore relativistic effects.

A. Perturbation Theory Results

Using first-order perturbation theory, we can write the transition matrix element between two states of definite photon number as

$$\langle m\psi_2|T|n\psi_1\rangle = -i \int_{t_0}^{t_1} dt' \langle m\psi_2|H'|n\psi_1\rangle e^{i\Delta E_{sy}t'}, \quad (14)$$

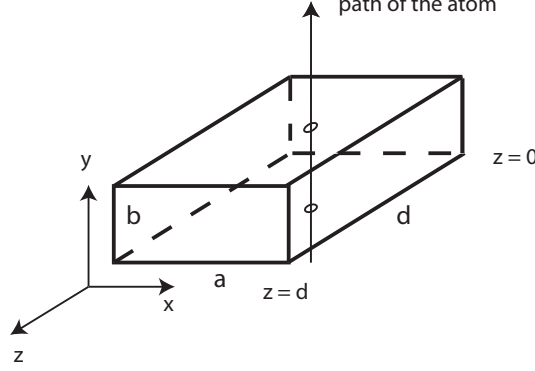


FIG. 1: A cavity for standing wave modes. We require that $b < a < d$, and $b \ll d$. The atom is assumed to pass through the cavity in the y -direction, as illustrated.

where ΔE_{sys} is the change in the energy of the atom-cavity system. The matrix element for the interaction Hamiltonian is

$$\begin{aligned} \langle m\psi_2|H'|n\psi_1\rangle &= -\langle\psi_2|d_y|\psi_1\rangle \langle m|\hat{E}_y|n\rangle \\ &= -\langle\psi_2|d_y|\psi_1\rangle (f\sqrt{n}\delta_{m,n-1} + f\sqrt{n+1}\delta_{m,n+1}). \end{aligned} \quad (15)$$

Here we assume that the electric field in the cavity is polarized in the y -direction.

The transition matrix element, Eq. (14) can then be written as

$$\langle m\psi_2|T|n\psi_1\rangle = i\langle\psi_2|d_y|\psi_1\rangle (f\sqrt{n}\delta_{m,n-1}I_1 + f\sqrt{n+1}\delta_{m,n+1}I_2), \quad (16)$$

where

$$I_1 = \int_{t_0}^{t_1} dt' e^{-i(\omega+\Delta\varepsilon)t'} = \frac{e^{-i(\omega+\Delta\varepsilon)t_1} - e^{-i(\omega+\Delta\varepsilon)t_0}}{-i(\omega + \Delta\varepsilon)}, \quad (17)$$

and

$$I_2 = \int_{t_0}^{t_1} dt' e^{i(\omega-\Delta\varepsilon)t'} = \frac{e^{i(\omega-\Delta\varepsilon)t_1} - e^{i(\omega-\Delta\varepsilon)t_0}}{i(\omega - \Delta\varepsilon)}. \quad (18)$$

Our Eq. (13), assuming a general one-mode initial photon state, Eq. (9), becomes

$$A_{2m} = i\langle\psi_2|d_y|\psi_1\rangle [\sqrt{m+1}c_{m+1}fI_1 + \sqrt{m}c_{m-1}fI_2]. \quad (19)$$

The probability, P_2 of finding the atom in the lower energy state $|\psi_2\rangle$ is

$$\begin{aligned} P_2 &= \sum_m |A_{2m}|^2 \\ &= |\langle\psi_2|d_y|\psi_1\rangle|^2 \sum_{m=0}^{\infty} \left[(m+1)|c_{m+1}|^2|f|^2|I_1|^2 + m|c_{m-1}|^2f^2|I_2|^2 \right. \\ &\quad \left. + 2\text{Re}\left(\sqrt{m(m+1)}f^2I_1^*I_2c_{m+1}^*c_{m-1}\right) \right]. \end{aligned} \quad (20)$$

We use the relations

$$\begin{aligned}
\sum_m (m+1) |c_{m+1}|^2 &= \langle n \rangle, \\
\sum_m m |c_{m-1}|^2 &= \langle n \rangle + 1, \\
\sum_m \sqrt{m(m+1)} c_{m-1} c_{m+1}^* &= \sum_n \sqrt{(n+1)(n+2)} c_n c_{n+2}^*.
\end{aligned} \tag{21}$$

Our expression for the probability is then

$$\begin{aligned}
P_2 &= |\langle \psi_2 | d_y | \psi_1 \rangle|^2 \left[\langle n \rangle f^2 |I_1|^2 + (\langle n \rangle + 1) f^2 |I_2|^2 \right. \\
&\quad \left. + 2 \operatorname{Re} \left(\sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} c_n c_{n+2}^* f^2 I_1^* I_2 \right) \right].
\end{aligned} \tag{22}$$

In the case when the initial state of the field is the vacuum state, $|\gamma_i\rangle = |0\rangle$, we have

$$P_2 = P_2(0) = |\langle \psi_2 | d_y | \psi_1 \rangle|^2 f^2 |I_2|^2. \tag{23}$$

It is important to note that this form of $P_2(0)$ assumes that the atom decays into only one mode. This should be a good approximation near resonance, but not otherwise. Let us now consider the ratio

$$\frac{P_2}{P_2(0)} = \langle n \rangle \left| \frac{I_1}{I_2} \right|^2 + \langle n \rangle + 1 + 2 \sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} \operatorname{Re} \left[c_n c_{n+2}^* \frac{I_2 I_1^*}{|I_2|^2} \right]. \tag{24}$$

From Eqs. (17) and (18), we have that

$$\left| \frac{I_1}{I_2} \right|^2 = \frac{1 - \cos[(\omega + \Delta\varepsilon)(t_1 - t_0)]}{1 - \cos[(\omega - \Delta\varepsilon)(t_1 - t_0)]} \frac{(\omega - \Delta\varepsilon)^2}{(\omega + \Delta\varepsilon)^2}. \tag{25}$$

In the limit when the transit time of the atom through the cavity is very short, $t_1 \rightarrow t_0$, and we have

$$\left| \frac{I_1}{I_2} \right|^2 \approx 1. \tag{26}$$

In the limit when $\Delta\varepsilon \rightarrow \omega$,

$$\begin{aligned}
\left| \frac{I_1}{I_2} \right|^2 &\approx \frac{1 - \cos 2\omega(t_1 - t_0)}{2\omega^2(t_1 - t_0)^2} \\
I_2 I_1^* &\approx -\frac{i}{2} \frac{(t_1 - t_0)}{\omega} (e^{2i\omega t_1} - e^{2i\omega t_0}) \\
|I_2|^2 &\approx (t_1 - t_0)^2,
\end{aligned} \tag{27}$$

and therefore we have that

$$\frac{I_2 I_1^*}{|I_2|^2} \approx -\frac{i e^{2i\omega t_0}}{2\omega(t_1 - t_0)} [e^{2i\omega(t_1 - t_0)} - 1]. \tag{28}$$

In addition to letting $\Delta\varepsilon \rightarrow \omega$, if we now also let $t_1 \rightarrow t_0$, the last equation above reduces to

$$\frac{I_2 I_1^*}{|I_2|^2} \approx -\frac{i e^{2i\omega t_0}}{2\omega (t_1 - t_0)} 2i\omega (t_1 - t_0) = e^{2i\omega t_0}. \quad (29)$$

Thus if we assume $(t_1 - t_0)\omega \ll 1$ and $(t_1 - t_0)\Delta\varepsilon \ll 1$, then

$$\left| \frac{I_1}{I_2} \right|^2 \approx 1. \quad (30)$$

If in addition, we take $\Delta\varepsilon \approx \omega$, we have

$$\frac{I_2 I_1^*}{|I_2|^2} \approx e^{2i\omega t_0}. \quad (31)$$

Therefore, the ratio of probabilities becomes

$$\frac{P_2}{P_2(0)} = 2\langle n \rangle + 1 + 2 \sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} \operatorname{Re}(c_n c_{n+2}^* e^{2i\omega t_0}). \quad (32)$$

Let us compare this with the expression for the expectation value, $\langle \hat{E}^2 \rangle$, of the normal-ordered squared electric field operator in the cavity. We can calculate this using the expansion of the operator in Eq. (5) and the state given in Eq. (9) for a single mode. We have

$$\begin{aligned} \langle E^2(\mathbf{x}_0, t) \rangle &= \langle \gamma_i | E^2(\mathbf{x}_0, t) | \gamma_i \rangle = f^2(\mathbf{x}_0) \sum_{nl} c_n^* c_l \langle n | 2a^\dagger a + a^2 e^{-2i\omega t} + (a^\dagger)^2 e^{2i\omega t} | l \rangle \\ &= f^2(\mathbf{x}_0) \sum_{nl} c_n^* c_l \left(2n \delta_{ln} + \sqrt{(n+2)(n+1)} \delta_{l, n+2} e^{-2i\omega t} \right. \\ &\quad \left. + \sqrt{n(n-1)} \delta_{l, n-2} e^{2i\omega t} \right) \\ &= f^2(\mathbf{x}_0) \sum_n \left[2n |c_n|^2 + \sqrt{(n+2)(n+1)} c_n^* c_{n+2} e^{-2i\omega t} \right. \\ &\quad \left. + \sqrt{n(n-1)} c_n^* c_{n-2} e^{2i\omega t} \right]. \end{aligned} \quad (33)$$

If we relabel $n \rightarrow n+2$, we can rewrite the last sum in the following way:

$$\begin{aligned} \sum_{n=2} \sqrt{n(n-1)} c_n^* c_{n-2} e^{2i\omega t} &= \sum_{n=0} \sqrt{(n+2)(n+1)} c_{n+2}^* c_n e^{2i\omega t} \\ &= \sum_{n=0} \sqrt{(n+2)(n+1)} (c_{n+2} c_n^* e^{-2i\omega t})^*. \end{aligned} \quad (34)$$

Using this and the fact that

$$\sum_{n=0} n |c_n|^2 = \langle n \rangle, \quad (35)$$

we may write Eq. (33) as

$$\langle E^2(\mathbf{x}_0, t) \rangle = f^2(\mathbf{x}_0) \left[2\langle n \rangle + 2 \sum_{n=0} \sqrt{(n+2)(n+1)} \operatorname{Re}(c_n c_{n+2}^* e^{2i\omega t}) \right]. \quad (36)$$

Near resonance, i.e., in the limit $\Delta\varepsilon \rightarrow \omega$, and in the limit of short transit times for the atom, i.e., $t_1 \rightarrow t_0$, the ratio of the de-excitation probabilities, $P_2/P_2(0)$, can therefore be written in terms of $\langle E^2(\mathbf{x}_0, t) \rangle$ as

$$\frac{P_2}{P_2(0)} = 1 + \frac{1}{f^2(\mathbf{x}_0)} \langle E^2(\mathbf{x}_0, t) \rangle. \quad (37)$$

Therefore we see that $P_2/P_2(0) < 1$ when $\langle E^2 \rangle < 0$. In the Appendix, we show that $\langle E^2 \rangle$ is bounded from below by $-f^2(\mathbf{x}_0)$. This guarantees that $P_2/P_2(0)$ is non-negative, as required.

Next we consider the case where we are near resonance, $\Delta\varepsilon \rightarrow \omega$, but with no restriction on $t_1 - t_0$. The ratio of probabilities is

$$\frac{P_2}{P_2(0)} = \langle n \rangle \left| \frac{I_1}{I_2} \right|^2 + \langle n \rangle + 1 + 2 \sum_n \sqrt{(n+1)(n+2)} \operatorname{Re} \left(c_n c_{n+2}^* \frac{I_1^* I_2}{|I_2|^2} \right), \quad (38)$$

where

$$\begin{aligned} \left| \frac{I_1}{I_2} \right|^2 &\approx \frac{1 - \cos 2\omega(t_1 - t_0)}{2\omega^2(t_1 - t_0)^2} \\ \frac{I_2 I_1^*}{|I_2|^2} &\approx -\frac{i e^{2i\omega t_0}}{2\omega(t_1 - t_0)} \left[e^{2i\omega(t_1 - t_0)} - 1 \right]. \end{aligned} \quad (39)$$

In this case, Eq. (38) becomes

$$\begin{aligned} \frac{P_2}{P_2(0)} &= \langle n \rangle + 1 + \langle n \rangle \frac{1 - \cos 2\omega(t_1 - t_0)}{2\omega^2(t_1 - t_0)^2} \\ &\quad - \sum_{n=0}^{\infty} \frac{\sqrt{(n+1)(n+2)}}{\omega(t_1 - t_0)} \operatorname{Re} \left[i c_n c_{n+2}^* e^{2i\omega t_0} \left(e^{2i\omega(t_1 - t_0)} - 1 \right) \right]. \end{aligned} \quad (40)$$

Note that $\langle E^2 \rangle$ contains the factor $e^{2i\omega t}$ and the integral of this expression,

$$\int_{t_0}^{t_1} dt e^{2i\omega t} = \frac{1}{2i\omega} (e^{2i\omega t_1} - e^{2i\omega t_0}) \quad (41)$$

does not have a factor of $1/(t_1 - t_0)$. Thus in general, $P_2/P_2(0)$ does not seem to be proportional to either $\langle E^2 \rangle$ or its time integral. When $\omega(t_1 - t_0) \gg 1$, we have

$$\frac{P_2}{P_2(0)} = \langle n \rangle + 1 + O\left(\frac{1}{\omega(t_1 - t_0)}\right) + \langle n \rangle O\left(\frac{1}{\omega^2(t_1 - t_0)^2}\right). \quad (42)$$

Therefore, in this limit $P_2/P_2(0) > 1$.

The advantage of considering the ratio $P_2/P_2(0)$ is that the precise form of the mode functions does not matter, since they cancel out. However, the drawback is that our expression, Eq. (37), is only valid near resonance since we assume decay into only one mode. To avoid this limitation, it is also useful to look at the difference, $\Delta P_2 = P_2 - P_2(0)$, so

that the contribution of the unexcited modes cancels, and hence we do not need to be near resonance. The difference, ΔP_2 , is given by

$$\Delta P_2 = |\langle \psi_2 | d_y | \psi_1 \rangle|^2 f^2 [\langle n \rangle (|I_1|^2 + |I_2|^2) + 2 \sum_n \sqrt{(n+1)(n+2)} \operatorname{Re}(c_n c_{n+2}^* I_1^* I_2)], \quad (43)$$

where

$$\begin{aligned} |I_1|^2 &= 2 \frac{1 - \cos(\omega + \Delta\varepsilon)(t_1 - t_0)}{(\omega + \Delta\varepsilon)^2}, \\ |I_2|^2 &= 2 \frac{1 - \cos(\omega - \Delta\varepsilon)(t_1 - t_0)}{(\omega - \Delta\varepsilon)^2}, \\ I_1^* I_2 &= \frac{e^{2i\omega t_1} + e^{2i\omega t_0} - e^{i[\omega(t_1+t_0)+\Delta\varepsilon(t_1-t_0)]} - e^{i[\omega(t_1+t_0)-\Delta\varepsilon(t_1-t_0)]}}{\Delta\varepsilon^2 - \omega^2}. \end{aligned} \quad (44)$$

We have assumed that the mode functions, f , are real. However this assumption is not really necessary, since one can always absorb the phase of the f 's into the phases of the complex coefficients, i.e., the c 's.

In the case where we are far below resonance, $\omega \ll \Delta\varepsilon$,

$$\begin{aligned} |I_1|^2 &\approx |I_2|^2 \approx \frac{2}{\Delta\varepsilon^2} [1 - \cos \Delta\varepsilon(t_1 - t_0)], \\ I_1^* I_2 &\approx \frac{[2 - e^{i\Delta\varepsilon(t_1-t_0)} - e^{-i\Delta\varepsilon(t_1-t_0)}]}{\Delta\varepsilon^2} \\ &\approx \frac{2}{\Delta\varepsilon^2} [1 - \cos \Delta\varepsilon(t_1 - t_0)]. \end{aligned} \quad (45)$$

This assumes that ωt_0 and ωt_1 are both much less than 1. Physically, we require that $\omega(t_1 - t_0) \ll 1$, and then for convenience set $t_0 = 0$. Using the above expressions, ΔP_2 becomes

$$\begin{aligned} \Delta P_2 &\approx 4 |\langle \psi_2 | d_y | \psi_1 \rangle|^2 f^2 \frac{[1 - \cos \Delta\varepsilon(t_1 - t_0)]}{\Delta\varepsilon^2} \times \left[\langle n \rangle + \sum_n \sqrt{(n+1)(n+2)} \operatorname{Re}(c_n c_{n+2}^*) \right] \\ &\approx 2 |\langle \psi_2 | d_y | \psi_1 \rangle|^2 \frac{[1 - \cos \Delta\varepsilon(t_1 - t_0)]}{\Delta\varepsilon^2} \langle \hat{E}^2(\mathbf{x}_0, t_0) \rangle. \end{aligned} \quad (46)$$

Thus we can have $\Delta P_2 < 0$, if $\langle E^2 \rangle < 0$ at the time that the atom transits the cavity. In the Appendix we show that $\langle E^2 \rangle$ is bounded from below and that, as a result, we have Eq. (A9)

$$\Delta P_2 \geq - \frac{2 |\langle \psi_2 | d_y | \psi_1 \rangle|^2 f^2(\mathbf{x}_0)}{\Delta\varepsilon^2}. \quad (47)$$

This result provides a limit on the degree to which sub-vacuum effects may suppress the decay probability. This limit is analogous to the quantum inequality bounds on negative energy densities and fluxes [4, 5, 6, 7, 8].

In the case where we are far above the resonant frequency, $\omega \gg \Delta\varepsilon$ and when $\Delta\varepsilon(t_1 - t_0) \ll 1$, we have

$$\begin{aligned} |I_1|^2 &\approx |I_2|^2 \approx \frac{2}{\omega^2} [1 - \cos \omega(t_1 - t_0)], \\ I_1^* I_2 &\approx - \frac{(e^{i\omega t_1} - e^{i\omega t_0})^2}{\omega^2}, \end{aligned} \quad (48)$$

and

$$\Delta P_2 \approx \frac{2}{\omega^2} |\langle \psi_2 | d_y | \psi_1 \rangle|^2 |f|^2 \left\{ 2\langle n \rangle (1 - \cos \omega(t_1 - t_0)) - \sum_n \sqrt{(n+1)(n+2)} \operatorname{Re} [c_n c_{n+2}^* (e^{i\omega t_1} - e^{i\omega t_0})^2] \right\}. \quad (49)$$

Although it is possible to have $\Delta P_2 < 0$, periods of $\Delta P_2 < 0$ do not seem to be correlated with either $\langle E^2 \rangle$ or its time integral in this case.

B. Specific quantum states

In this subsection, we will discuss two specific non-classical states of the photon field.

1. Vacuum plus two photon state

One of the simplest examples of a quantum state which exhibits negative energy density and $\langle E^2 \rangle < 0$ is a coherent superposition of the vacuum and a state containing two photons in the same mode. Such a state can be expressed as

$$|\gamma\rangle = \frac{1}{\sqrt{1+\beta^2}} (|0\rangle + \beta|2\rangle), \quad (50)$$

where we take β to be a real, non-negative parameter. In this state, the mean squared electric field may be written as

$$\langle E^2(\mathbf{x}_0, t) \rangle = \frac{2\beta}{1+\beta^2} f^2(\mathbf{x}_0) [2\beta + \sqrt{2} \cos(2\omega t)], \quad (51)$$

which reaches its minimum value when $\cos(2\omega t) = -1$. If $\beta < \sqrt{2}/2$, then this value is negative. At this point, the ratio of probabilities may be expressed as

$$\frac{P_2}{P_2(0)} = 1 + \frac{2\beta}{1+\beta^2} (2\beta + \sqrt{2}). \quad (52)$$

This ratio is plotted in Fig. 2, from which we can see that $P_2/P_2(0)$ reaches its minimum value of about 0.55 when $\beta \approx 0.32$. Thus it is possible to achieve a 45% reduction in the decay probability in this state.

The mean squared electric field is plotted as a function of time in Fig. 3 for the case $\beta = 0.32$. Note that $\langle E^2 \rangle < 0$ about 1/3 of the time. However, because $\langle E^2 \rangle$ oscillates at an angular frequency of 2ω , the duration of the interval when $\langle E^2 \rangle < 0$ is approximately $\Delta t \approx \pi/(3\omega) \approx 1/\omega$.

2. Squeezed vacuum state

Another example of a quantum state which exhibits sub-vacuum effects is the squeezed vacuum state, described by the complex parameter $\zeta = r e^{i\phi}$. This state has been studied

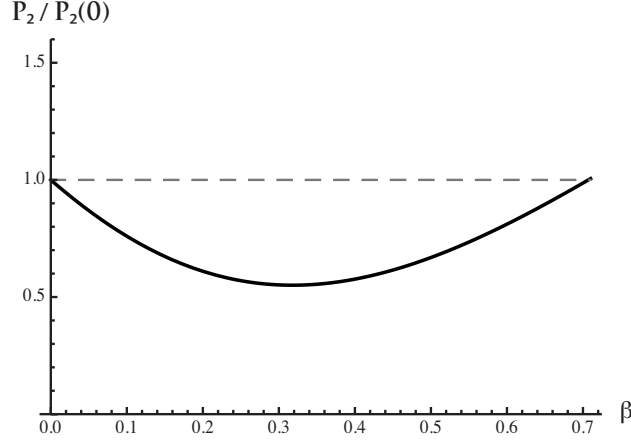


FIG. 2: The ratio of the decay probability in the vacuum plus two photon state to that in the vacuum state is plotted as a function of the parameter β .

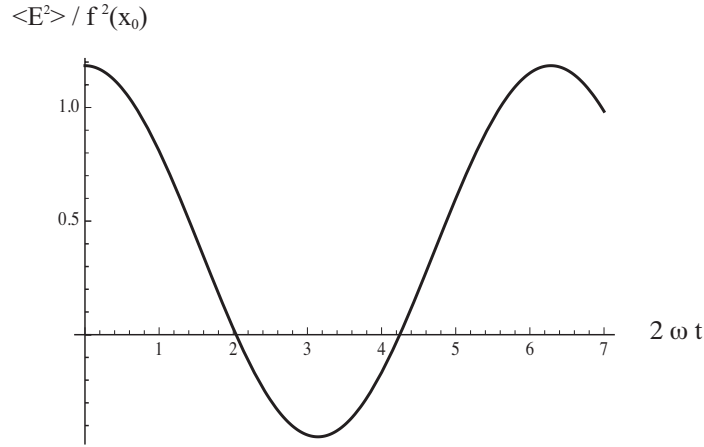


FIG. 3: The mean squared electric field in the vacuum plus two photon state is plotted as a function of time for the case $\beta = 0.32$.

extensively, beginning with the work of Caves [20]. Using the results in Ref. [20], one may show that the mean squared electric field in a squeezed vacuum state becomes

$$\langle E^2(\mathbf{x}_0, t) \rangle = 2f^2(\mathbf{x}_0) [\sinh^2 r + \cosh r \sinh r \cos(\phi + 2\omega t)]. \quad (53)$$

This quantity is most negative when $\cos(\phi + 2\omega t) = -1$, at which point we have

$$\langle E^2(\mathbf{x}_0, t) \rangle = -f^2(\mathbf{x}_0) (1 - e^{-2r}). \quad (54)$$

If we compare this relation with Eq. (37), we see that $P_2 \rightarrow 0$ for $r \gg 1$. Thus, in the limit of large squeeze parameter, the decay rate can momentarily go close to zero. It is also of interest to note that in this limit, the inequality Eq. (47) becomes an equality, as may be seen from Eqs. (46) and (A8).

C. Standing wave modes

To make the time interval when $\langle E^2 \rangle < 0$ as long as possible, we are interested in the lowest frequency mode of the cavity depicted in Fig. 1. A straightforward calculation using the formalism of Chapter 8 of Jackson [21] shows that, with our condition that $b < a < d$, the lowest frequency mode is the TE mode with $p = l = 1, m = 0$, where

$$\omega = \pi \sqrt{\frac{1}{a^2} + \frac{1}{d^2}}, \quad (55)$$

and

$$\begin{aligned} E_x &= E_z = 0, \\ E_y &= \frac{\omega a}{\pi} A_{10} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{d}z\right), \\ B_x &= i\frac{a}{d} A_{10} \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{\pi}{d}z\right), \\ B_y &= 0, \\ B_z &= -iA_{10} \cos\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{d}z\right), \end{aligned} \quad (56)$$

where the electric field is taken to be polarized in the y -direction. Here A_{10} is a real normalization constant, which we will now determine.

In Lorentz-Heaviside units, we can write the energy density for a classical electromagnetic field as

$$\begin{aligned} \rho &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) = \frac{1}{2}(E_y^2 + |B_x|^2 + |B_z|^2) \\ &= \frac{1}{2}A_{10}^2 \left[\frac{\omega^2 a^2}{\pi^2} \sin^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{d}z\right) + \frac{a^2}{d^2} \sin^2\left(\frac{\pi}{a}x\right) \cos^2\left(\frac{\pi}{d}z\right) \right. \\ &\quad \left. + \cos^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{d}z\right) \right], \end{aligned} \quad (57)$$

where we have used the expressions given in Eq. (56) above. Let us normalize the vacuum mode functions by setting

$$\int d^3x \rho = \int_0^a dx \int_0^b dy \int_0^d dz \rho = \frac{1}{2}\omega. \quad (58)$$

This leads to the result

$$A_{10}^2 = \frac{2\omega}{a b d \left(1 + \frac{a^2}{d^2}\right)}, \quad (59)$$

and

$$E_y^2 = \frac{2\omega^3 a}{\pi^2 b d \left(1 + \frac{a^2}{d^2}\right)} \sin^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{d}z\right). \quad (60)$$

D. Difference in probabilities for low ω

We may use the above results to give expressions for the quantity ΔP_2 in the case $\omega \ll \Delta\varepsilon$. Combining Eq. (46), and Eqs. (60) and (55) for the $p = l = 1, m = 0$, TE mode, we have

$$\Delta P_2 \approx 8\pi \left(1 + \frac{a^2}{d^2}\right)^{1/2} \frac{|\langle \psi_2 | d_y | \psi_1 \rangle|^2}{a^2 b d (\Delta\varepsilon)^2} \sin^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{d}z\right)$$

$$\times [1 - \cos \Delta \varepsilon(t_1 - t_0)] \times \left[\langle n \rangle + \sum_n \sqrt{(n+1)(n+2)} \operatorname{Re}(c_n c_{n+2}^*) \right]. \quad (61)$$

E. Does $\langle E^2 \rangle < 0$ imply $\langle \rho \rangle < 0$?

Ideally one would like our system to be such that $\langle E^2 \rangle < 0$ when $\langle \rho \rangle < 0$, so that $\langle E^2 \rangle$ tracks negative energy density. For a classical single plane wave mode, $E^2 = B^2$, in Lorentz-Heaviside units. Therefore, for the quantized field, $\langle E^2 \rangle = \langle B^2 \rangle$, and $\rho = (\langle E^2 \rangle + \langle B^2 \rangle)/2$ imply that $\langle E^2 \rangle < 0$ when $\langle \rho \rangle < 0$. Unfortunately, for modes in a cavity the situation is more complicated, and in general $\langle E^2 \rangle \neq \langle B^2 \rangle$. We may rewrite Eq. (5) as

$$\mathbf{E} = \sum_j (a_j \mathbf{E}_j + a_j^\dagger \mathbf{E}_j^*), \quad (62)$$

and the corresponding expression for the magnetic field operator as

$$\mathbf{B} = \sum_j (a_j \mathbf{B}_j + a_j^\dagger \mathbf{B}_j^*). \quad (63)$$

Here

$$\mathbf{E}_j = \hat{\mathbf{e}}_j f_j(\mathbf{x}) e^{-i\omega t} \quad (64)$$

and

$$\mathbf{B}_j = \frac{i}{\omega} \hat{\mathbf{e}}_j \times \nabla f_j(\mathbf{x}) e^{-i\omega t}. \quad (65)$$

The latter expression follows from the Maxwell equation $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$, and the vector identity $\nabla \times (\hat{\mathbf{e}} f) = \hat{\mathbf{e}} \times \nabla f$ for a constant vector $\hat{\mathbf{e}}$.

For the case where only a single mode j is excited, the normal-ordered expectation values of the squared fields are

$$\langle E^2 \rangle = 2\langle a^\dagger a \rangle |\mathbf{E}_j|^2 + 2\operatorname{Re}(\langle a^2 \rangle \mathbf{E}_j^2) \quad (66)$$

and

$$\langle B^2 \rangle = 2\langle a^\dagger a \rangle |\mathbf{B}_j|^2 + 2\operatorname{Re}(\langle a^2 \rangle \mathbf{B}_j^2). \quad (67)$$

For a general mode function, the second terms in the expressions for $\langle E^2 \rangle$ and $\langle B^2 \rangle$ are not in phase.

For example, consider our situation of interest, a single TE mode in a rectangular cavity, with $p = l = 1, m = 0$. For this mode we have, from Eq. (56), that

$$\begin{aligned} \mathbf{E}_j^2 &= E_y^2 = \frac{\omega^2 a^2}{\pi^2} A_{10}^2 \sin^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{d}z\right) e^{-2i\omega t} \\ \mathbf{B}_j^2 &= B_x^2 + B_z^2 = -A_{10}^2 \left[\cos^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{d}z\right) + \frac{a^2}{d^2} \sin^2\left(\frac{\pi}{a}x\right) \cos^2\left(\frac{\pi}{d}z\right) \right] e^{-2i\omega t}. \end{aligned} \quad (68)$$

These two expressions have opposite signs, so when $\langle E^2 \rangle < 0$, $\langle B^2 \rangle > 0$, and vice versa. Therefore, periods when $\langle E^2 \rangle < 0$ do not, in general, necessarily correspond to periods of negative energy density.

However, there are special cases when $|\langle E^2 \rangle| \gg |\langle B^2 \rangle|$ and $\rho \approx \langle E^2 \rangle$. For the mode discussed above, this occurs when $x \approx a/2$ and either $z \approx d/2$ or $a \ll d$. In these cases, experiments which measure $\langle E^2 \rangle$ are also detecting the mean energy density, ρ .

F. Comparison with the Model of Ford, Grove, and Ottewill

In this subsection, we will compare some aspects of our model with the spin model of Ford, Grove, and Ottewill [10], which was summarized in Sect I. Ford *et al* assumed plane wave modes, so $\rho = \langle E^2 \rangle$, whereas we use cavity modes for which $\rho \neq \langle E^2 \rangle$, except in special cases. Another difference is that the response of the spin system, measured by a quantity analogous to our ΔP_2 , seems to track the energy density whenever the system is far from resonance, either $\omega \gg \Omega$ or $\omega \ll \Omega$, where ω is the frequency of the excited mode of the photon field and Ω is the resonant frequency of the spin system. By contrast, we found a correlation of $\langle \hat{E}^2 \rangle$ with $P_2/P_2(0)$ only near resonance, and a correlation with ΔP_2 only far below resonance. This difference can be traced to the use of adiabatic switching in Ref. [10]. At $t = t_0$, the spin system is assumed to be in the lower energy state, and then one takes the limit $t_0 \rightarrow -\infty$, and averages oscillating quantities. In this limit, for example $\sin^2[(1/2)(\Omega - \omega)(t - t_0)] \rightarrow 1/2$, if $\Omega \neq \omega$. This is used to go from Eq. (4.8) to Eq. (4.9) in Ref. [10]. After this is done, it is not possible to let $\omega \rightarrow \Omega$. For example, the quantity

$$\frac{\sin^2[(1/2)(\Omega - \omega)(t - t_0)]}{(\Omega - \omega)^2} \rightarrow \begin{cases} \frac{1}{4}(t - t_0)^2, & \text{for } \omega \rightarrow \Omega, \text{ with } t_0 \text{ fixed, as in our case} \\ \frac{1}{2}(\Omega - \omega)^{-2}, & \text{adiabatic switching.} \end{cases} \quad (69)$$

This describes a system which was prepared in the lower state and then coupled to the radiation field in the distant past. In this case, one obtains expressions such as Eq. (4.12) in Ref. [10], with factors of $1/(\Omega - \omega)^2$, $1/(\Omega + \omega)^2$, and $1/(\Omega^2 - \omega^2)$, which are proportional to $\langle E^2 \rangle$ (or to the energy density, ρ), only if $\omega \gg \Omega$ or $\omega \ll \Omega$.

The adiabatic switching assumption may be appropriate for a system coupled to photons in a plane wave mode, but does not apply to our model. The entrance of the atoms into a cavity is more accurately described by the sudden approximation employed here.

IV. SOME NUMERICAL ESTIMATES

A. Criteria to be Fulfilled

In this section, we will analyze the feasibility of observing the suppression of the de-excitation probability. We consider an atom passing through a cavity with a single mode excited. Let us first list the desired criteria for a measurement of P_2 . We wish to impose the following conditions:

- (1) $\omega \approx \Delta\varepsilon$. This will allow us to apply Eq. (37), and have a situation where the de-excitation probability tracks the mean squared electric field.
- (2) It will be convenient to consider Rydberg atoms, which are often used in cavity QED experiments. Their transition frequencies are typically in the microwave range. Consider the specific case of the transition between the $n = 51$ and the $n = 50$ energy levels, with a transition frequency of $f = 51.1$ GHz and wavelength $\lambda \approx 6$ mm [19].
- (3) The excited cavity mode is the lowest mode, $\omega = \pi\sqrt{1/a^2 + 1/d^2}$, so

$$f_{\text{cavity}} = \frac{\omega}{2\pi} = \frac{c}{2} \sqrt{\frac{1}{a^2} + \frac{1}{d^2}}. \quad (70)$$

This requires the two longer dimensions of the cavity, a and d , each to be at least 3 mm.

- (4) For the atom to fit in the cavity, we need the smallest dimension of the cavity, b , to be larger than the size of the atom. The size of a Rydberg atom with $n = 50$ is about 100 nm. So we need

$$b > 100 \text{ nm} = 0.1 \mu\text{m}. \quad (71)$$

- (5) We want the travel time for the atom to traverse the cavity dimension b , to be no more than the time interval in which $\langle \hat{E}^2 \rangle < 0$, which is a fraction of the period of the cavity mode.

- (6) To avoid the complications of relativity, and so that the atom sees a small Doppler shift of the cavity frequency, we want

$$\frac{v}{c} \ll 1. \quad (72)$$

- (7) We would ideally like the lifetime, τ , of the excited state to be roughly equal to the transit time of the atom across the cavity. For the Rydberg atoms under consideration ($n = 50$), $\tau \approx 3.6 \times 10^{-2}$ sec. This would make $P_2(0)$ of order unity and maximize the effect we are studying.

B. Order-of-Magnitude Estimates

Let us consider the case of the vacuum plus two photon state discussed in Sect. IIIB 1. Such a state might be created by a process which has a finite probability to emit a pair of photons into the lowest mode of the cavity. We have seen that in this state it is possible to have $P_2/P_2(0) \approx 0.55$, so there is a significant reduction in the decay rate. The duration of the interval of negative mean squared electric field is, from the discussion at the end of Sect. IIIB 1,

$$\Delta t \approx \frac{1}{6f} \approx 3 \times 10^{-11} \text{ s}. \quad (73)$$

Let us choose the smallest dimension of the cavity to be

$$b \approx 1 \mu\text{m}, \quad (74)$$

which easily satisfies criterion (4). This requires that the speed of the atom be

$$v > \frac{b}{\Delta t} \approx 3 \times 10^5 \text{ m/s}. \quad (75)$$

This can be satisfied by non-relativistic speeds compatible with criterion (6).

The one criterion listed above which cannot be satisfied is (7), as the atomic lifetime is necessarily long compared to the transit time. This means that each atom has a probability of less than 10^{-9} of decaying while transiting the cavity. However, this need not be a serious problem if a sufficiently large flux of atoms can be used.

V. SUMMARY AND DISCUSSION

In this paper, we have considered the effects of vacuum fluctuation suppression on the de-excitation probabilities of atoms. This suppression occurs in quantum states of the radiation field in which the renormalized expectation value of the square of the electric field operator is periodically negative. Examples of such states include the vacuum plus two photon state and the single-mode squeezed vacuum state. Both of these are examples of quantum states exhibiting sub-vacuum effects, such as negative energy density or negative mean squared electric field.

We have treated a model detector which consists of an atom in an excited state traversing a cavity containing photons in a non-classical state. We calculated the ratio, $P_2/P_2(0)$, of the de-excitation probability in an arbitrary single-mode cavity field quantum state to the same probability in the vacuum state. This calculation assumed that the atom and field are near resonance, with the transition frequency approximately equal to the lowest cavity frequency, so that the atom had only one available decay mode. Our results showed that near resonance, the ratio $P_2/P_2(0)$ tracks $\langle E^2 \rangle$. Hence $P_2/P_2(0)$ will have its minimum value when $\langle E^2 \rangle$ is maximally negative. In certain cases, this is also when the energy density is maximally negative.

We also calculated the difference, $\Delta P_2 = P_2 - P_2(0)$, of the de-excitation probability between our arbitrary state and the vacuum state, where the effects of the non-excited modes would cancel out. In this case $\Delta P_2 = P_2 - P_2(0)$ is proportional to $\langle E^2 \rangle$ far below resonance, but not in other cases. In particular, only when the lowest cavity mode frequency was much less than the transition frequency of the atom did we find that $\Delta P_2 < 0$ when $\langle E^2 \rangle < 0$. In this case, we were able to derive a quantum inequality lower bound on ΔP_2 .

Finally, we discussed the plausibility of an experiment involving Rydberg atoms to detect $\langle E^2 \rangle < 0$ and possibly negative energy density. Although challenging, such an experiment may be possible.

Acknowledgments

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APPENDIX A: BOUND ON ΔP_2

Here we will derive a quantum inequality bound on the amount by which the de-excitation probability may be suppressed by quantum field effects. From Eq. (36), we have

$$\langle \gamma_i | \hat{E}^2(\mathbf{x}_0, t) | \gamma_i \rangle = f^2(\mathbf{x}_0) \left[2 \langle n \rangle + 2 \sum_{n=0} \sqrt{(n+2)(n+1)} \operatorname{Re}(c_n c_{n+2}^* e^{2i\omega t}) \right]. \quad (\text{A1})$$

Let us examine the term in brackets. We can always choose the phases of the c 's so that the magnitude of the second term will be largest at $t = 0$. Let us call the smallest possible value of the bracketed term S , i.e.,

$$S = 2 \langle n \rangle + 2 \sum_{n=0} \sqrt{(n+2)(n+1)} \operatorname{Re}(c_n c_{n+2}^*). \quad (\text{A2})$$

Following the argument given on p. 230 of Ref.[4], we will prove that S , and thus $\langle E^2 \rangle$, are bounded from below. Since S also appears in our expressions, Eq. (37) and Eq. (46), we can show that these quantities are bounded below as well. Using the fact that

$$\langle n \rangle = \sum_{n=0}^{\infty} n |c_n|^2, \quad (\text{A3})$$

we may expand S as follows,

$$S = 2 \sum_{n=0}^{\infty} n |c_n|^2 + \sum_{n=0}^{\infty} \sqrt{(n+2)(n+1)} c_n c_{n+2}^* + \sum_{n=0}^{\infty} \sqrt{(n+2)(n+1)} c_{n+2}^* c_n. \quad (\text{A4})$$

Rewriting the right-hand side of Eq. (A3), we have

$$\begin{aligned} \sum_{n=0}^{\infty} n |c_n|^2 &= \sum_{n=2}^{\infty} (n-2) |c_{n-2}|^2 \\ &= \sum_{n=1}^{\infty} (n-1) |c_{n-2}|^2 - \sum_{n=2}^{\infty} |c_{n-2}|^2 \\ &= -1 + \sum_{n=1}^{\infty} (n-1) |c_{n-2}|^2. \end{aligned} \quad (\text{A5})$$

Note also that

$$\begin{aligned} \sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} c_n c_{n+2}^* &= \sum_{\ell=2}^{\infty} \sqrt{\ell-1} \sqrt{\ell} c_{\ell-2} c_{\ell}^* \\ &= \sum_{n=1}^{\infty} \sqrt{n(n-1)} c_{n-2} c_n^*. \end{aligned} \quad (\text{A6})$$

To get the first equality, we let $n+2 \rightarrow \ell$; to get the second equality, we relabeled $\ell \rightarrow n$, and used the fact that the $n=1$ term does not contribute anything to the sum. If we now substitute these expressions into Eq. (A4), and relabel appropriately, we can write S as

$$\begin{aligned} S &= -1 + \sum_{n=1}^{\infty} \left[n |c_n|^2 + (n-1) |c_{n-2}|^2 + \sqrt{n(n-1)} c_{n-2} c_n^* + \sqrt{n(n-1)} c_n^* c_{n-2} \right] \\ &= -1 + \sum_{n=1}^{\infty} \left| \sqrt{n} c_n + \sqrt{n-1} c_{n-2} \right|^2. \end{aligned} \quad (\text{A7})$$

Hence $S \geq -1$, and so we have that

$$\langle E^2 \rangle \geq -f^2(\mathbf{x}_0), \quad (\text{A8})$$

and when $\omega \ll \varepsilon$, from Eq. (46),

$$\Delta P_2 \geq -\frac{2 |\langle \psi_2 | d_y | \psi_1 \rangle|^2 f^2(\mathbf{x}_0)}{\Delta \varepsilon^2}. \quad (\text{A9})$$

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